Classical & Quantum Waves

Waves in dispersive media (review)

- Last lecture, we considered what happens w/ 2 waves in a dispersive media, where the wave velocity is frequency dependent.

$$\Psi = \Psi_1 + \Psi_2 = \underbrace{A\cos(k_1 x - \omega_1 t)}_{\Psi_1} + \underbrace{A\cos(k_2 x - \omega_2 t)}_{\Psi_2}$$

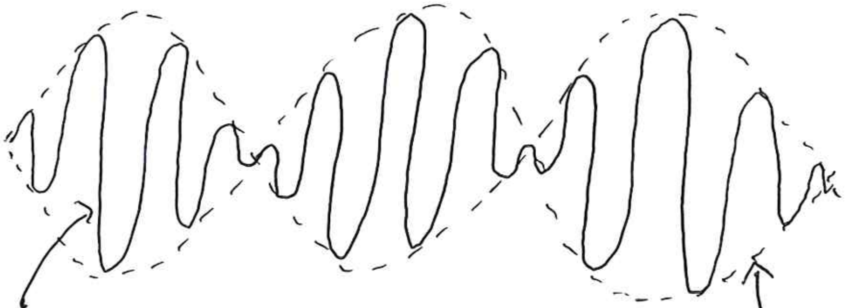
- We showed, after some trig identies, that the above can be rewritten as:

$$\Psi = \underbrace{2A\cos(x\Delta k - t\Delta\omega)}_{\equiv A(x,t)}\cos(k_0 x - \omega_0 t)$$

$$\text{w/} \quad k_0 \equiv \frac{k_1 + k_2}{2} \qquad \omega_0 \equiv \frac{\omega_1 + \omega_2}{2}$$

$$\Delta k \equiv \frac{k_2 - k_1}{2} \qquad \Delta\omega \equiv \frac{\omega_2 - \omega_1}{2}$$

- This describes a wave oscillating w/ avg. frequency ω_0 and avg. wavenumber k_0, that is amplitude modulated at a frequency $\Delta\omega$ and wavenumber Δk.



Fast oscillation is wave w/ ω_0, k_0

slow modulation / envelope is wave w/ $\Delta\omega$, Δk

- The fast oscillation, which is sometimes called the "carrier wave", travels at the "phase velocity" $\boxed{v_{ph} = \frac{\omega_0}{k_0}}$ [sometimes we just write v for v_{ph}]

- The modulation travels at the "group velocity", which is defined as $\boxed{v_g = \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0}}$

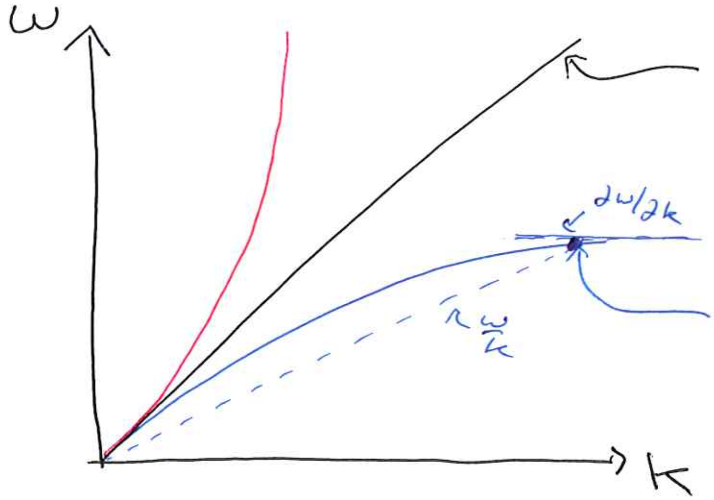
- In non-dispersive media, where $v_1 = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = v_2$, $v_{ph} = \frac{\omega_0}{k_0} = v_g = \left.\frac{\partial \omega}{\partial k}\right|_{k_0}$

  → means that the wave shape drawn above moves together at same speed

- In dispersive media, $v_1 \neq v_2$ & $v_{ph} \neq v_g$. This means that the carrier wave & modulation travel at different speeds!

Dispersion relation

- This describes how the frequency of a wave depends on the wavenumber. It also encodes the phase and group velocities.



straight line means no dispersion.

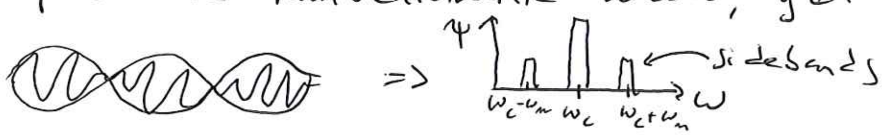
$V = \frac{\omega}{k} = \frac{\partial \omega}{\partial k} = v_g$ for all ω, k

downward bending means $v_{ph} = \frac{\omega}{k}$ is greater than $v_g = \frac{\partial \omega}{\partial k}$. You can see this b/c slope of ω/k is > slope of $\frac{\partial \omega}{\partial k}$ at point drawn.

- upward bending means $v_g > v_{ph}$

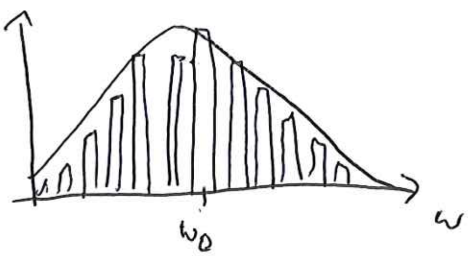
Wave packets

- When superimpose 2 monochromatic waves, get a pattern of beats

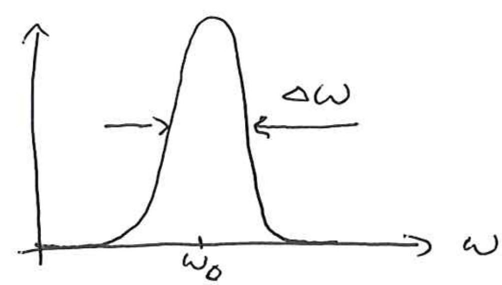


- Can also group together many different waves of different frequencies, and in most physical situations this is usually the case.

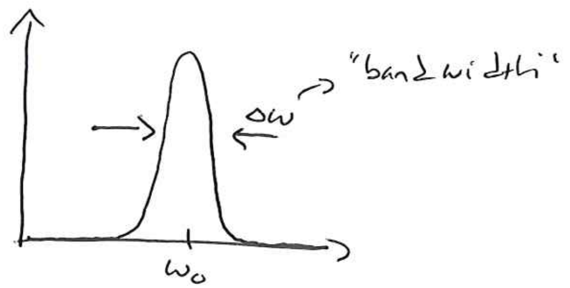
- If add more modulations to a wave, can get more sidebands in frequency domain



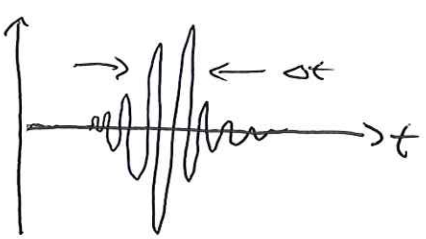
limit of continuous frequency distribution



· In limit of continuous frequency distribution, one can construct a frequency distribution that is peaked around a central frequency ω_0, w/ a distribution width $\Delta\omega$ that is small compared to ω_0



↓ Fourier transform to time-domain



· In the time-domain, this frequency distribution gives a short pulse of waves

↳ "wave packet"

· Wavepacket travels at the group velocity $v_g = \frac{d\omega}{dk}$

· We will see that the widths of the distribution in frequency $\Delta\omega$ and time Δt obey the "bandwidth theorem": $\Delta\omega \, \Delta t \gtrsim 2\pi$

→ important for classical & quantum waves!

· Note ~~that~~ inverse relation b/t $\Delta\omega$ and Δt

→ long pulse in time means narrow in frequency

→ short pulse in time means broad in frequency

Examples:

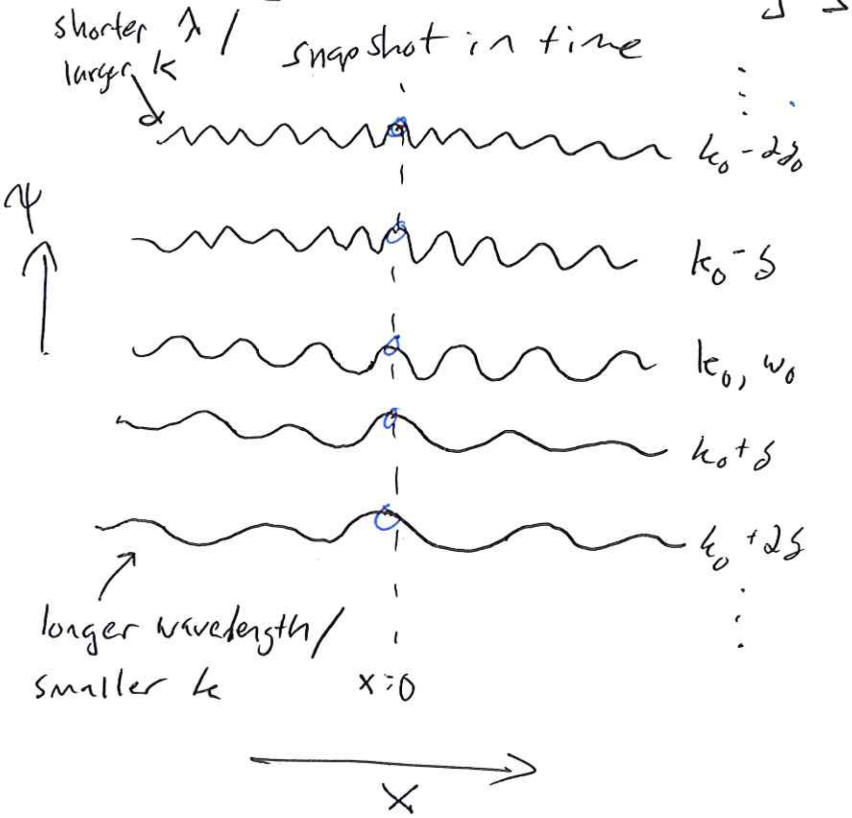
- Short pulses of light used in optical communication to take advantage of broader frequency bandwidth (where information is encoded)
- Femtosecond (10^{-15} s) laser pulses used for machining and fundamental scientific research (Nobel prize 2018, 2023)
- QUANTUM WAVES . . .

Formulation of a wave packet

- Consider superposition of a group of monochromatic waves w/ discrete wavenumbers
- Each wave has $\psi_n = a_n \cos(k_n x - \omega_n t)$, their superposition is:

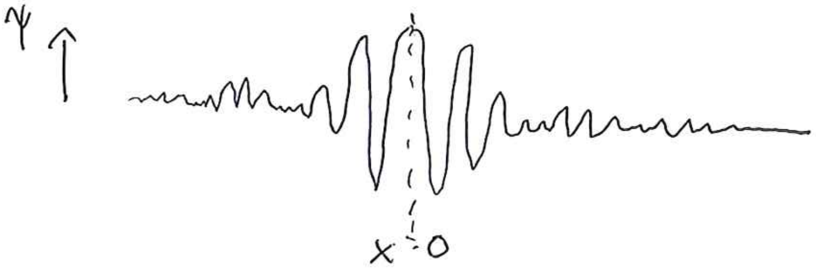
$$\psi = \sum_n \psi_n = \sum_n a_n \cos(k_n x - \omega_n t)$$

[See Figure 8.6 in King]



- At $x=0$, all waves are in phase at their max value.
- As go away from $x=0$, waves slip out of phase b/c slightly different k, ω

· If add all these waves together, they constructively interfere at $x=0$, and (mostly) destructively interfere away from $x=0$



· Our eq. $\psi = \sum a_n \cos(k_n x - \omega_n t)$ describing the superposition can be rewritten as:

$$\psi = A(x,t)\cos(k_0 x - \omega_0 t)$$

w/ k_0, ω_0 central values of pulse

$$\text{w/ } A(x,t) = a\,\frac{\sin[n(x\delta k - t\delta\omega)/2]}{\sin[(x\delta k - t\delta\omega)/2]}$$

[no proof]

where n is # of waves in the group

· Just like our case of 2 monochromatic waves, the carrier wave travels at phase velocity $v_{ph} = \frac{\omega_0}{k_0}$ and the "modulation" = wavepacket travels at group velocity $v_g = \frac{d\omega}{dk}$

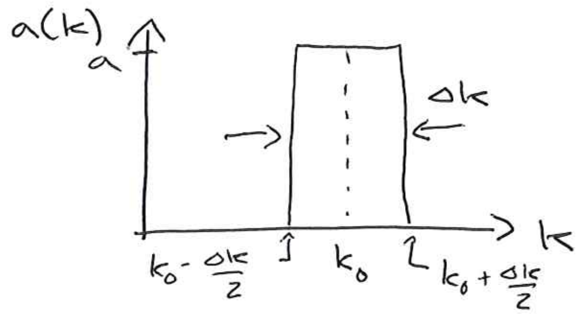
- Now consider a <u>continuous</u> distribution of wavenumbers centered at k_0 w/ width $\Delta k \ll k_0$

- The sum Σ from the discrete case considered earlier gets replaced with an integral:

$$\psi = \int a(k) \cos(kx - \omega t)\, dk$$

w/

$$a(k) = \begin{cases} a, & \text{if } |k - k_0| \leq \frac{\Delta k}{2} \\ 0, & \text{if } |k - k_0| > \frac{\Delta k}{2} \end{cases}$$



- This assumption for $a(k)$ in this example allows us to evaluate ψ:

$$\psi = a \int_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}} \cos(kx - \omega t)\, dk$$

- Recall that ω is k-dependent $\omega(k)$. We can account for this dependence by using a Taylor expansion, since we assume that the range of wavenumbers Δk is small compared to k_0

$$\omega = \omega_0 + \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0} (k-k_0) + \dots \qquad \omega_0 = \omega(k_0)$$

let $\alpha \equiv \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0}$ (to simplify things)

- plugging the Taylor expanded ω into the argument of our cosine function gives

$$kx - \omega t = kx - [\omega_0 + \alpha(k-k_0)]t = k(x-\alpha t) - (\omega_0 - \alpha k_0)t$$

$$= \underbrace{k(x-\alpha t) - \beta t}_{\equiv \xi} \quad w/ \ \beta \equiv \omega_0 - \alpha k_0$$

$\rightarrow$ use this as new variable for integration

$$\rightarrow \frac{d\xi}{dk} = (x-\alpha t) \Rightarrow d\xi = (x-\alpha t)\,dk$$

$$dk = \frac{d\xi}{(x-\alpha t)}$$

- rewrite integral as

$$\psi = a \int_{\xi_1}^{\xi_2} \frac{\cos\xi \, d\xi}{(x-\alpha t)}$$

$$w/ \ \xi_1 = \left(k_0 - \frac{\Delta k}{2}\right)(x-\alpha t) - \beta t$$

$$\xi_2 = \left(k_0 + \frac{\Delta k}{2}\right)(x-\alpha t) - \beta t$$

[no proof, but can show using definitions above w/ some algebra]

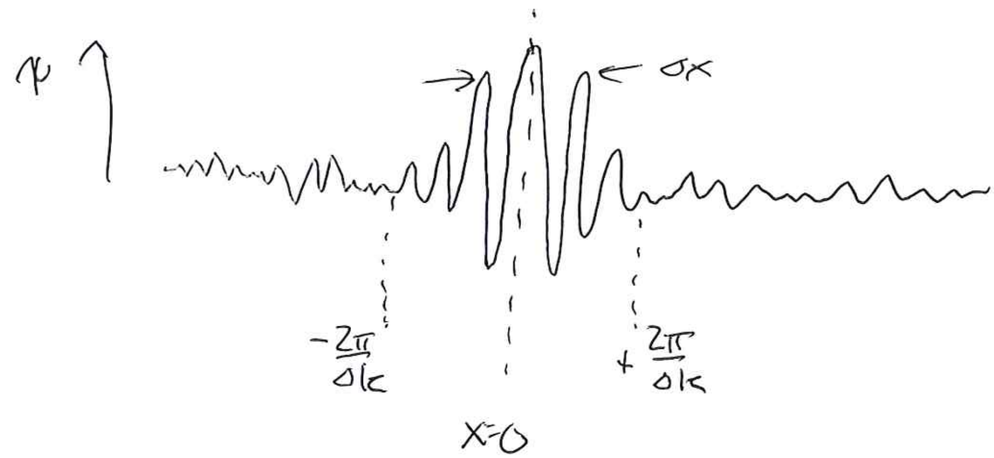
- This is a manageable integral, get:

$$\psi = \frac{a}{(x-\alpha t)} \underbrace{(\sin \xi_2 - \sin \xi_1)}$$

$$= 2\sin\left[(\xi_2 - \xi_1)/2\right]\cos\left[(\xi_2+\xi_1)/2\right]$$

$$\Rightarrow \psi = A(x,t)\cos(k_0 x - \omega_0 t)$$

w/ $$A(x,t) = a\Delta k \frac{\sin[\Delta k(x-\alpha t)/2]}{\Delta k(x-\alpha t)/2}$$



wave packet!

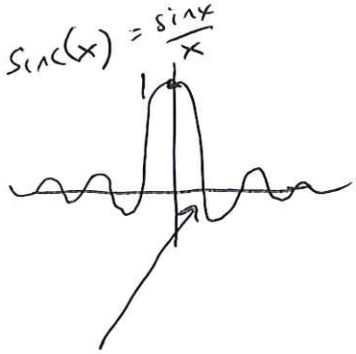
- $$V_g = \frac{dx}{dt} = \alpha \equiv \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0}$$

- Important note: we assumed $\Delta k \ll k_0$ so only keep linear term in Taylor expansion. Under this condition, the wavepacket retains its shape while propagating
  → otherwise need nonlinear terms which causes packet to spread out as propagating

- At a given snapshot in time, say $t=0$, get

$$A(x) = a\Delta k \underbrace{\frac{\sin(x\Delta k/2)}{x\Delta k/2}}_{\text{Sinc Function}}$$

"Shape" of the envelope that moves



→ at $x=0$, has value $=1$

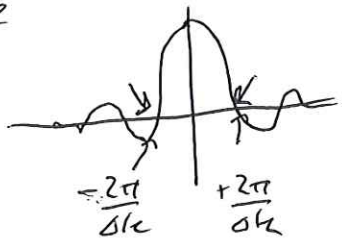
1st zero occurs at $\frac{x\Delta k}{2} = \pm\pi$

→ This fact gives us the relation $x = \pm\frac{2\pi}{\Delta k}$

w/ Δk the width of the wavenumber distribution

- How to define width of wavepacket in space, Δx?

- Could take the first zeros on positive and negative side



. But more common to use $\frac{1}{2}$ this value ⇒ $\Delta x \approx \frac{2\pi}{\Delta k}$

→ point is, this is kind of an ~~[illegible]~~ unexact procedure

→ gives relation $\boxed{\Delta x \Delta k \approx 2\pi}$ "Bandwidth theorem"

~~→ same concept as $\Delta t \Delta\omega \approx 2\pi$ we showed earlier~~

- This expression shows that the shorter the length of a wavepacket, the greater the range of wavenumbers needed to represent it
- For monochromatic wave, $\Delta k = 0$ and $\Delta x = \infty$
- Can also express bandwidth theorem in terms of Frequency & time.

$$\Delta t = \frac{\Delta x}{d\omega/dk} \Rightarrow \Delta t \Delta \omega = \Delta x \Delta k$$

$$v_g = \frac{\Delta \omega}{\Delta k} \qquad \Rightarrow \boxed{\Delta t \Delta \omega \approx 2\pi}$$

- Bandwidth theorem closely related to "Heisenberg uncertainty principle" from quantum mechanics, where particles are described as waves ~~packets~~.
- Position of particle is "somewhere" within region Δx.
- The wavelength of particle is given by de Broglie: $\lambda = \frac{h}{p}$

  * w/ h: Planck's constant, p: momentum
- Since $\lambda = \frac{2\pi}{k}$ $\Rightarrow \boxed{\Delta x \Delta p \approx \hbar}$

  → Shows that the more you are certain about position of particle, the less you are certain about momentum. Purely a wave phenomenon!